COMMUTING ISOMETRIES AND JOINT INVARIANT SUBSPACES

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Abstract. An n-isometry, $n \geq 2$, is an n-tuple commuting isometries $(V_1, \ldots, V_n)$ on a Hilbert space $\mathcal{H}$ such that if V is a shift, where

$$V = \prod_{i=1}^{n} V_i.$$

In this paper we provide an analytic representations of n-isometries. Also we present a description of joint invariant subspaces for n-isometries.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space. Let $(V_1, \ldots, V_n)$ be an n-tuple of commuting isometries on $\mathcal{H}$. In this note, we always assume that $n \geq 2$ is a positive integer. A closed subspace $\mathcal{S} \subseteq \mathcal{H}$ is said to be joint invariant for $(V_1, \ldots, V_n)$ if $V_i \mathcal{S} \subseteq \mathcal{S}$, $i = 1, \ldots, n$. We say that $(V_1, \ldots, V_n)$ is an *n-isometry* if V is a shift, where

$$V = \prod_{i=1}^{n} V_i.$$

Recall that an isometry X on $\mathcal{H}$ is said to be a *shift* if $X^{*m} \to 0$ as $m \to \infty$ in the strong operator topology or, equivalently, if X on $\mathcal{H}$ has no unitary summand. Moreover, if X is a shift, then X on $\mathcal{H}$ and M_z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$ are unitarily equivalent, where $\mathcal{W}(X) = \ker X^*$ and $H^2_{\mathcal{W}(X)}(\mathbb{D})$ is the $\mathcal{W}(X)$-valued Hardy space and M_z is the multiplication operator by the coordinate function z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$ (see Section 2).

In this paper we aim to address two basic issues of n-isometries: (i) analytic and canonical models for n-isometries, and (ii) classification of joint invariant subspaces for n-isometries. To that aim, we consider the initial approach by Berger, Coburn and Lebow [3] from a more modern point of view (due to Bercovici, Douglas and Foias [2]). In our approach we will also follow the recent paper [11].

Our first main result, Theorem 2.1, states that if $(V_1, \ldots, V_n)$ is an n-isometry on a Hilbert space $\mathcal{H}$, then $(V_1, \ldots, V_n)$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ are unitarily equivalent, where $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a *canonical* model n-isometry on some vector-valued Hardy space $H^2_{\mathcal{W}}(\mathbb{D})$. The model n-isometries are defined as follows (see Bercovici, Douglas and Foias [2]). Consider a Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$, and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$. Let $\{\Phi_1, \ldots, \Phi_n\} \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions (polynomials) on $\mathbb{D}$, where

$$\Phi_i(z) = U_i(P_i^{\perp} + zP_i) \qquad (z \in \mathbb{D}),$$

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and $i = 1, \ldots, n$. Then the n-tuple of multiplication operators $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ is called a *model n-isometry* if the following conditions are satisfied:

(a) $U_i U_j = U_j U_i$ for all $i, j = 1, \ldots n$;
(b) $U_1 \cdots U_n = I_{\mathcal{E}}$;
(c) $P_i + U_i^* P_j U_i = P_j + U_j^* P_i U_j \leq I_{\mathcal{E}}$ for all $i \neq j$; and
(d) $P_1 + U_1^* P_2 U_1 + U_1^* U_2^* P_3 U_2 U_1 + \cdots + U_1^* U_2^* \cdots U_{n-1}^* P_n U_{n-1} \cdots U_2 U_1 = I_{\mathcal{E}}$.

Throughout the paper, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ will denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{S}$. We also set

$$P_{\mathcal{S}}^{\perp} = I_{\mathcal{H}} - P_{\mathcal{S}}.$$

In [2], motivated by Berger, Coburn and Lebow [3], Bercovici, Douglas and Foias proved the following result: An n-isometry is unitarily equivalent to a model n-isometry. Equivalently, given an n-isometry $(V_1, \ldots, V_n)$ on $\mathcal{H}$, one can solve the above equations (a)-(d) for some Hilbert space $\mathcal{E}$, unitary operators $\{U_1, \ldots, U_n\}$ on $\mathcal{E}$, and orthogonal projections $\{P_1, \ldots, P_n\}$ on $\mathcal{E}$. Here, in Theorem 2.1, we give an explicit and canonical solution to above problem. This also gives a new proof of Bercovici, Douglas and Foias theorem.

On the one hand, our model n-isometry is explicit and canonical. On the other hand, our proof is perhaps more computational and less conceptual than the one in [2]. Another advantage of our approach is the proof of a list of useful equalities related to commuting isometries, which can be useful in other contexts.

Our second main result concerns a characterization of joint invariant subspaces of model n-isometries. To be precise, let $\mathcal{W}$ be a Hilbert space, and let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$. Let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. In Theorem 4.1, we prove that $\mathcal{S}$ is a joint $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ invariant subspace if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ (the Beurling-Lax-Halmos inner multiplier corresponding to the shift invariant subspace $\mathcal{S}$ of $H^2_{\mathcal{W}_*}(\mathbb{D})$) and a model n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_i \Theta = \Theta \Psi_i,$$

for all $i = 1, \ldots, n$.

The paper is organized as follows. In Section 2 we study and review the analytic construction of n-isometries. In Section 3 we study more closely at the n-isometries and examine a canonical (or model) n-isometry. The proof of the invariant subspace theorem is contained in Section 4.

2. n-isometries

In this section, following [11], we derive an explicit analytic representation of n-isometries. For motivation, let us recall that if X on $\mathcal{H}$ is an isometry, then X is a shift operator if and only if X and M_z on $H^2_{\mathcal{W}(X)}(\mathbb{D})$ are unitarily equivalent. Explicitly, if X is a shift on $\mathcal{H}$, then

$$\mathcal{H} = \bigoplus_{m=0}^{\infty} X^m \mathcal{W}(X),$$

where

$$\mathcal{W}(X) = \ker X^* = \mathcal{H} \ominus X\mathcal{H},$$

is the *wandering subspace* for X (see Halmos [7]). Hence the natural map $\Pi_X : \mathcal{H} \to H^2_{\mathcal{W}(X)}(\mathbb{D})$ defined by

$$\Pi_X(X^m \eta) = z^m \eta,$$

for all $m \geq 0$ and $\eta \in \mathcal{W}(X)$, is a unitary operator and

$$\Pi_X X = M_z \Pi_X.$$

We call Π_X the *Wold-von Neumann decomposition* of the shift X (see [11]).

Now let $(V_1, \ldots, V_n)$ be an n-isometry. We will use the following notations:

$$V = \prod_{i=1}^{n} V_i,$$

and

$$\tilde{V}_i = \prod_{j \neq i} V_i,$$

and for simplicity of notation we set

$$\mathcal{W} = \mathcal{W}(V),$$

and

$$\tilde{\mathcal{W}}_i = \mathcal{W}(\tilde{V}_i),$$

for all $i = 1, \ldots, n$. Clearly

$$\tilde{V}_n = V.$$

Let $\Pi_V : \mathcal{H} \to H^2_{\mathcal{W}}(\mathbb{D})$ be the Wold-von Neumann decomposition of V. Then

$$\Pi_V V_i \Pi_V^* \in \{M_z\}',$$

and hence there exists $\Phi_i \in H^\infty_{\mathcal{B}(\mathcal{W})}(\mathbb{D})$ such that

$$\Pi_V V_i = M_{\Phi_i} \Pi_V,$$

for all $i = 1, \ldots, n$. We now proceed to compute the bounded analytic functions $\{\Phi_j\}_{j=1}^n$. Our method follows the construction in [11]. In fact, a close variant of Theorem 2.1 below follows from Theorems 3.4 and 3.5 of [11]. We will only sketch the construction, highlighting the essential ingredients for our present purpose.

Let $j \in \{1, \ldots, n\}$, $w \in \mathbb{D}$ and $\eta \in \mathcal{W}$. Then

$$\begin{aligned} \Phi_j(w)\eta &= (M_{\Phi_j}\eta)(w) \\ &= (\Pi_V V_j \Pi_V \eta)(w) \\ &= (\Pi_V V_j \eta)(w), \end{aligned}$$

But

$$I_{\mathcal{H}} = P_{\tilde{\mathcal{W}}_j} + \tilde{V}_j \tilde{V}_j^*,$$

yields that

$$V_j \eta = V_j P_{\tilde{\mathcal{W}}_j} \eta + V \tilde{V}_j^* \eta,$$

and thus

$$\begin{aligned}\Pi_V V_j \eta &= \Pi_V (V_j P_{\tilde{\mathcal{W}}_j} \eta + V \tilde{V}_j^* \eta) \\ &= \Pi_V (V_j P_{\tilde{\mathcal{W}}_j} \eta) + \Pi_V (V \tilde{V}_j^* \eta) \\ &= V_j P_{\tilde{\mathcal{W}}_j} \eta + M_z \tilde{V}_j^* \eta,\end{aligned}$$

as $\Pi_V V = M_z \Pi_V$ and $V^*(V_j(I - \tilde{V}_j \tilde{V}_j^*) V_j^*) = 0$. Therefore, it follows that

$$\Phi_j(w)\eta = V_j P_{\tilde{\mathcal{W}}_j} \eta + w \tilde{V}_j^* \eta.$$

Since $\mathcal{W} = \tilde{V}_j \mathcal{W}_j \oplus \tilde{\mathcal{W}}_j$, we deduce that

$$\Phi_j(w) = V_j|_{\tilde{\mathcal{W}}_j} + w \tilde{V}_j^*|_{\tilde{V}_j \mathcal{W}_j}.$$

Finally, $\mathcal{W} = \mathcal{W}_j \oplus V_j \tilde{\mathcal{W}}_j$ implies that

$$U_j = \begin{bmatrix} \tilde{V}_j^*|_{\tilde{V}_j \mathcal{W}_j} & 0 \\ 0 & V_j|_{\tilde{\mathcal{W}}_j} \end{bmatrix} : \begin{matrix} \tilde{V}_j \mathcal{W}_j \\ \oplus \\ \tilde{\mathcal{W}}_j \end{matrix} \to \begin{matrix} \mathcal{W}_j \\ \oplus \\ V_j \tilde{\mathcal{W}}_j \end{matrix},$$

is a unitary operator on $\mathcal{W}$. Therefore

$$\Phi_j(w) = U_j (P_{\tilde{\mathcal{W}}_j} + w P_{\tilde{\mathcal{W}}_j}^{\perp}) \qquad (w \in \mathbb{D}).$$

Note that it follows from the definition of U_j that

$$U_j = (V_j P_{\tilde{\mathcal{W}}_j} + \tilde{V_j}^*)|_{\mathcal{W}}.$$

This and

$$V_j P_{\tilde{\mathcal{W}}_j} = P_{\mathcal{W}} V_j, \tag{2.1}$$

yields

$$U_j = (P_{\mathcal{W}} V_j + \tilde{V_j}^*)|_{\mathcal{W}}.$$

Summarizing the discussion above, we have the following:

Theorem 2.1. *Let $(V_1, \ldots, V_n)$ be an n-isometry on a Hilbert space $\mathcal{H}$. If $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is the representation of $(V_1, \ldots, V_n)$, then*

$$\Phi_j(z) = U_j (P_{\tilde{\mathcal{W}}_j} + z P_{\tilde{\mathcal{W}}_j}^{\perp})|_{\mathcal{W}},$$

for all $z \in \mathbb{D}$, where

$$U_j = (P_{\mathcal{W}} V_j + \tilde{V_j}^*)|_{\mathcal{W}}.$$

is a unitary operator on $\mathcal{W}$ and $j = 1, \ldots, n$.

In the following section, we will explore the coefficients of Φ_j, $j = 1, \ldots, n$, in more details.

3. Model n-isometries

In this section, we propose a canonical model for n-isometries. We study the coefficients of the one-variable polynomials in Theorem 2.1 more closely and prove that the corresponding n-isometry $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ is a model n-isometry (see Section 1 for the definition). We again point out that the above assertion follows from Bercovici, Douglas and Foias [2] and our presentation below is more explicit and influenced by the refinements from [11].

Let $(V_1, \ldots, V_n)$ be an n-isometry on a Hilbert space $\mathcal{H}$. Consider the analytic representation $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{W}}(\mathbb{D})$ of $(V_1, \ldots, V_n)$ as in Theorem 2.1. First we prove that $\{U_j\}_{j=1}^n$ is a commutative family. Let $p, q \in \{1, \ldots, n\}$ and $p \neq q$. As $\mathcal{W} = \ker V^*$, it follows that

$$\tilde{V}_p^* \tilde{V}_q^*|_{\mathcal{W}} = 0.$$

Then using (2.1) we obtain

$$\begin{aligned} U_p U_q &= (P_{\mathcal{W}} V_p + \tilde{V}_p^*)(P_{\mathcal{W}} V_q + \tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}} V_p P_{\mathcal{W}} V_q + \tilde{V}_p^* P_{\mathcal{W}} V_q + P_{\mathcal{W}} V_p \tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}} V_p V_q + \prod_{i \neq p,q} V_i^* P_{\tilde{\mathcal{W}}_q} + V_p P_{\tilde{\mathcal{W}}_p} \tilde{V}_q^*)|_{\mathcal{W}} \\ &= (P_{\mathcal{W}} V_p V_q + (\prod_{i \neq p,q} V_i^*)(P_{\tilde{\mathcal{W}}_q} + \tilde{V}_q P_{\tilde{\mathcal{W}}_p} \tilde{V}_q^*))|_{\mathcal{W}} \\ &= (P_{\mathcal{W}} V_p V_q + (\prod_{i \neq p,q} V_i^*))|_{\mathcal{W}}, \end{aligned}$$

as $(\tilde{V}_q P_{\tilde{\mathcal{W}}_p} \tilde{V}_q^* + P_{\tilde{\mathcal{W}}_q})|_{\mathcal{W}} = I_{\mathcal{W}}$, and hence

$$U_p U_q = U_q U_p,$$

follows by duality. Now if $I \subseteq \{1, \ldots, n\}$, then the same line of arguments, as above, along with the fact that

$$P_{\mathcal{W}}(\prod_{i \in I} V_i) P_{\mathcal{W}} = P_{\mathcal{W}}(\prod_{i \in I} V_i),$$

yields

$$\prod_{i \in I} U_i = (P_{\mathcal{W}}(\prod_{i \in I} V_i) + (\prod_{i \in I^c} V_i^*))|_{\mathcal{W}}. \tag{3.1}$$

In particular, since $P_{\mathcal{W}} V|_{\mathcal{W}} = 0$, we have that

$$\prod_{i=1}^n U_i = I_{\mathcal{W}}.$$

The following lemma may be of independent interest.

Lemma 3.1. *Fix $1 \leq j \leq n$. Let $I \subseteq \{1, \ldots, n\}$, and let $j \in I$. Then*

$$(\prod_{i \in I} U_i) P^{\perp}_{\tilde{\mathcal{W}}_j} (\prod_{i \in I} U_i^*) = (\prod_{i \in I \setminus \{j\}} V_i) P_{\mathcal{W}_j} (\prod_{i \in I \setminus \{j\}} V_i^*)|_{\mathcal{W}}.$$

Proof. Note that $\mathcal{W} \ominus \tilde{\mathcal{W}}_j = \tilde{V}_j \mathcal{W}_j \subseteq \mathcal{W}$. Then

$$P^{\perp}_{\tilde{\mathcal{W}}_j} = \tilde{V}_j (I - V_j V_j^*) \tilde{V}_j^*.$$

By once again using the fact that $V^*|_{\mathcal{W}} = P_{\mathcal{W}}V|_{\mathcal{W}} = 0$, and by (3.1), one sees that

$$\begin{aligned}
(\prod_{i\in I} U_i)P^{\perp}_{\tilde{\mathcal{W}}_j}(\prod_{i\in I} U_i^*) &= [P_{\mathcal{W}}(\prod_{i\in I} V_i) + (\prod_{i\in I^c} V_i^*)]\tilde{V}_j(I - V_jV_j^*)\tilde{V}_j^*[(\prod_{i\in I} V_i^*) + P_{\mathcal{W}}(\prod_{i\in I^c} V_i)]|_{\mathcal{W}}\\
&= (\prod_{i\in I^c} V_i^*)P^{\perp}_{\tilde{\mathcal{W}}_j}P_{\mathcal{W}}(\prod_{i\in I^c} V_i)|_{\mathcal{W}}\\
&= (\prod_{i\in I^c} V_i^*)P^{\perp}_{\tilde{\mathcal{W}}_j}(\prod_{i\in I^c} V_i)|_{\mathcal{W}}\\
&= (\prod_{i\in I^c} V_i^*)\tilde{V}_j(I - V_jV_j^*)\tilde{V}_j^*(\prod_{i\in I^c} V_i)|_{\mathcal{W}}\\
&= (\prod_{i\in I\setminus\{j\}} V_i)P_{\mathcal{W}_j}(\prod_{i\in I\setminus\{j\}} V_i^*)|_{\mathcal{W}}.
\end{aligned}$$

This completes the proof of the lemma. ■

Let $p, q \in \{1, \dots, n\}$, and let $p \neq q$. A computation similar to the proof of the above lemma yields that

$$U_p^*P^{\perp}_{\tilde{\mathcal{W}}_q}U_p = (\prod_{i\neq p,q} V_i)P_{\mathcal{W}_q}(\prod_{i\neq p,q} V_i^*).$$

Then

$$\begin{aligned}
(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^*P^{\perp}_{\tilde{\mathcal{W}}_q}U_p) &= [\tilde{V}_p(I - V_pV_p^*)\tilde{V}_p^* + (\prod_{i\neq p,q} V_i)P_{\mathcal{W}_q}(\prod_{i\neq p,q} V_i^*)]|_{\mathcal{W}}\\
&= (\prod_{i\neq p,q} V_i)(V_qV_q^* + P_{\mathcal{W}_q})(\prod_{i\neq p,q} V_i^*)|_{\mathcal{W}}\\
&= (\prod_{i\neq p,q} V_i)(\prod_{i\neq p,q} V_i^*)|_{\mathcal{W}}\\
&= P_{\mathcal{W}}P_{\tilde{\mathcal{W}}_{p,q}}|_{\mathcal{W}},
\end{aligned}$$

where $\tilde{\mathcal{W}}_{p,q} = \text{ran}\,(\prod_{i\neq p,q} V_i)$. Therefore

$$(P^{\perp}_{\tilde{\mathcal{W}}_p} + U_p^*P^{\perp}_{\tilde{\mathcal{W}}_q}U_p) = (P^{\perp}_{\tilde{\mathcal{W}}_q} + U_q^*P^{\perp}_{\tilde{\mathcal{W}}_p}U_q) \leq I_{\mathcal{W}}.$$

Finally, let $1 \leq j \leq n-1$ and $I_j = \{j, \dots, n-1\}$. Then Lemma 3.1 implies

$$(\prod_{i\in I_j} U_i)P^{\perp}_{\tilde{\mathcal{W}}_j}(\prod_{i\in I_j} U_i^*) = [(\prod_{i\in I_{j+1}} V_i)(\prod_{i\in I_{j+1}} V_i^*) - (\prod_{i\in I_j} V_i)(\prod_{i\in I_j} V_i^*)]|_{\mathcal{W}}.$$

This and

$$P^{\perp}_{\tilde{\mathcal{W}}_n} = \tilde{V}_n\tilde{V}_n^* - VV^*,$$

implies that

$$\sum_{j=1}^{n-1}(\prod_{i\in I_j} U_i)P^{\perp}_{\tilde{\mathcal{W}}_j}(\prod_{i\in I_j} U_i^*) + P^{\perp}_{\tilde{\mathcal{W}}_n} = (I - VV^*)|_{\mathcal{W}},$$

that is

$$\sum_{j=1}^{n-1}(\prod_{i\in I_j} U_i)P^{\perp}_{\tilde{\mathcal{W}}_j}(\prod_{i\in I_j} U_i^*) + P^{\perp}_{\tilde{\mathcal{W}}_n} = I_{\mathcal{W}}.$$

Multiplying both sides by $\prod_{i=1}^{n-1} U_i$ on the right and $\prod_{i=1}^{n-1} U_i^*$ on the left gives

$$P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^*P^{\perp}_{\tilde{\mathcal{W}}_2}U_1 + U_1^*U_2^*P^{\perp}_{\tilde{\mathcal{W}}_2}U_2U_1 + \cdots + (\prod_{i=1}^{n-1} U_i^*)P^{\perp}_{\tilde{\mathcal{W}}_n}(\prod_{i=1}^{n-1} U_i) = I_{\mathcal{W}}.$$

We summarize the above as follows.

Theorem 3.2. *If $(V_1, \ldots, V_n)$ be an n-isometry on a Hilbert space $\mathcal{H}$, then*
(a) $U_p U_q = U_q U_p$ *for* $p, q = 1, \ldots n$,
(b) $\prod_{p=1}^n U_p = I_{\mathcal{W}}$,
(c) $(P^{\perp}_{\tilde{\mathcal{W}}_i} + U_i^* P^{\perp}_{\tilde{\mathcal{W}}_j} U_i) = (P^{\perp}_{\tilde{\mathcal{W}}_j} + U_j^* P^{\perp}_{\tilde{\mathcal{W}}_i} U_j) \leq I_{\mathcal{W}}$,
(d) $P^{\perp}_{\tilde{\mathcal{W}}_1} + U_1^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_1 + U_1^* U_2^* P^{\perp}_{\tilde{\mathcal{W}}_2} U_2 U_1 + \cdots + (\Pi_{i=1}^{n-1} U_i^*) P^{\perp}_{\tilde{\mathcal{W}}_n} (\Pi_{i=1}^{n-1} U_i) = I_{\mathcal{W}}$.

As a corollary, we have:

Corollary 3.3. *Let $\mathcal{H}$ be a Hilbert space and $(V_1, \ldots, V_n)$ be an n-isometry on $\mathcal{H}$. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be the n-isometry as constructed in Theorem 2.1, and let $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, unitary operators $\{\tilde{U}_i\}_{i=1}^n$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\tilde{\mathcal{W}}$, be a model n-isometry. Then:*
(a) $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model n-isometry.
(b) $(V_1, \ldots, V_n)$ and $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ are unitarily equivalent.
(b) $(V_1, \ldots, V_n)$ and $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ are unitarily equivalent if and only if there exists a unitary operator $W : \mathcal{W} \to \tilde{\mathcal{W}}$ such that $WU_i = \tilde{U}_i W$ and $WP_i = \tilde{P}_i W$ for all $i = 1, \ldots, n$.

Proof. The first part is a direct consequence of the previous theorem. The second part is easy and readily follows from Theorem 4.1 in [11] or Theorem 2.9 in [2]. ∎

Combining this corollary with Theorem 3.2, we have the following characterization of commutative isometric factors of shift operators.

Corollary 3.4. *Let $\mathcal{E}$ be a Hilbert space, and let $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ be a commutative family of isometric multipliers. Then*

$$M_z = \prod_{i=1}^n M_{\Phi_j},$$

or, equivalently

$$\prod_{i=1}^n \Phi_j = zI_{\mathcal{E}},$$

if and only if, up to unitary equivalence, $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is a model n-isometry.

In other words, $zI_{\mathcal{E}}$ factors as n commuting isometric multipliers $\{\Phi_i\}_{i=1}^n \subseteq H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ if and only if there exist unitary operators $\{U_i\}_{i=1}^n$ on $\mathcal{E}$ and orthogonal projections $\{P_i\}_{i=1}^n$ on $\mathcal{E}$ satisfying the properties (a) - (d) in Theorem 3.2 such that $\Phi_i(z) = U_i(P_i^{\perp} + zP_i)$ for all $i = 1, \ldots, n$.

4. Joint Invariant Subspaces

Let $\mathcal{W}$ be a Hilbert space. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a model n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed joint $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ invariant subspace of $H^2_{\mathcal{W}}(\mathbb{D})$, that is

$$M_{\Phi_i} \mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Then $(M_{\Phi_1}|_{\mathcal{S}}, \ldots, M_{\Phi_n}|_{\mathcal{S}})$ is an n-isometry on $\mathcal{S}$, and hence by Corollary 3.3, there exists a model n-isometry $(M_{\Psi_1}, \ldots, M_{\Psi_n})$ on $H^2_{\tilde{\mathcal{W}}}(\mathbb{D})$, for some Hilbert space $\tilde{\mathcal{W}}$, such that $(V_1|_{\mathcal{S}}, \dots, V_n|_{\mathcal{S}})$ and $(M_{\Psi_1}, \dots, M_{\Psi_n})$ are unitarily equivalent. The main purpose of this section is to describe the joint invariant subspaces for $(M_{\Phi_1}, \dots, M_{\Phi_n})$ in terms of the model n-isometry $(M_{\Psi_1}, \dots, M_{\Psi_n})$.

As a motivational example, consider the classical $n = 1$ case. Here the model 1-isometry is a shift operator M_z on $H^2_{\mathcal{W}}(\mathbb{D})$ for some Hilbert space $\mathcal{W}$. Let $\mathcal{S}$ be a non-trivial closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then by the Beurling [4], Lax [9] and Halmos [7] theorem, $\mathcal{S}$ is shift invariant if and only if there exist a Hilbert space $\mathcal{W}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}).$$

Moreover, in this case, if we set

$$V = M_z|_{\mathcal{S}},$$

then V on $\mathcal{S}$ and M_z on $H^2_{\mathcal{W}_*}(\mathbb{D})$ are unitarily equivalent. This follows directly from the above representation of $\mathcal{S}$. Indeed, it follows that $X = M_\Theta : H^2_{\mathcal{W}_*}(\mathbb{D}) \to \text{ran} M_\Theta = \mathcal{S}$ is a unitary operator and

$$X M_z = V X.$$

Turning to the case $n > 1$, let $\mathcal{S}$ be a closed invariant subspace of $H^2_{\mathcal{W}}(\mathbb{D})$ and let $(M_{\Phi_1}, \dots, M_{\Phi_n})$ be a model n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$. From

$$\prod_{j=1}^n M_{\Phi_j} = M_z,$$

we see that $\mathcal{S}$ is a shift invariant subspace of $H^2_{\mathcal{W}}(\mathbb{D})$ and therefore by Beurling, Lax and Halmos theorem, there exist a Hilbert space $\mathcal{W}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*, \mathcal{W})}(\mathbb{D})$ such that $\mathcal{S}$ can be represented as

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

If $1 \le j \le n$, then $\Phi_j \mathcal{S} \subseteq \mathcal{S}$ implies that

$$\text{ran } (M_{\Phi_j} M_\Theta) \subseteq \text{ran } M_\Theta,$$

and so by Douglas's range and inclusion theorem

$$\Phi_j \Theta = \Theta \Psi_j,$$

for some $\Psi_j \in H^\infty_{\mathcal{B}(\mathcal{W}_*)}(\mathbb{D})$. Note that $M_{\Phi_j} M_\Theta$ is an isometry and $\|\Theta \Psi_j f\| = \|\Psi_j f\|$ for each $f \in H^2_{\mathcal{W}_*}(\mathbb{D})$. But then

$$\|\Psi_j f\| = \|f\|,$$

and it follows that Ψ_j is an inner function, and hence $M_{\Psi_j} = M_\Theta^* M_{\Phi_j} M_\Theta$. So

$$\prod_{i=1}^n M_{\Psi_i} = \prod_{i=1}^n (M_\Theta^* M_{\Phi_i} M_\Theta),$$

Now

$$M_\Theta M_\Theta^* M_{\Phi_j} M_\Theta = M_{\Phi_j} M_\Theta.$$

as $\Phi_j\Theta H^2_{\mathcal{W}_*}(\mathbb{D}) \subseteq H^2_{\mathcal{W}}(\mathbb{D})$. Consequently

$$\begin{aligned}\prod_{j=1}^{n} M_{\Psi_j} &= M_\Theta^*(\prod_{j=1}^{n} M_{\Phi_j})M_\Theta^* \\ &= M_\Theta^* M_z M_\Theta \\ &= M_\Theta^* M_\Theta M_z \\ &= M_z,\end{aligned}$$

that is, $(M_{\Psi_1},\dots,M_{\Psi_n})$ is an n-isometry on $H^2_{\mathcal{W}_*}(\mathbb{D})$. Therefore, we have the following theorem:

Theorem 4.1. *Let $\mathcal{W}$ be a Hilbert space. Let $(M_{\Phi_1},\dots,M_{\Phi_n})$ be an n-isometry on $H^2_{\mathcal{W}}(\mathbb{D})$, and let $\mathcal{S}$ be a closed subspace of $H^2_{\mathcal{W}}(\mathbb{D})$. Then $\mathcal{S}$ is a joint $(M_{\Phi_1},\dots,M_{\Phi_n})$ invariant subspace if and only if there exist a Hilbert space $\mathcal{W}_*$, an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}_*,\mathcal{W})}(\mathbb{D})$ and an n-isometry $(M_{\Psi_1},\dots,M_{\Psi_n})$ on $H^2_{\mathcal{W}_*}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{W}_*}(\mathbb{D}),$$

and

$$\Phi_j\Theta = \Theta\Psi_j,$$

for all $j = 1,\dots,n$.

It is curious to note that the content of Theorem 4.1 is related to the question [1] and its answer [15] on the classifications of invariant subspaces of Γ-isometries. We also refer to the recent paper [10] for a representation of shift invariant subspaces of the Hardy space over unit polydisc.

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